

MAT436 ESSAY: THE SPECTRAL THEOREM

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1. INTRODUCTION

Professor Elliott summarizes the contents of the class of MAT436 with just two words - Spectrum, Index. I have made up my mind to embrace these ideas. This essay attempts to circulate around the idea of spectrum, which also appears in many different contexts, including algebraic geometry. (Don't get fooled by the terminology "spectra" in algebraic topology! What does stabilizing spaces have to do with this anyways!) This mainly follows the wonderful notes of David Ben-Zvi [Ben26], [Deb21] and [Dyk21]. The purpose is to reinterpret some of the things we have discussed in class, in the perspective relevant to the Geometric Langlands Program.

Remark 1.0.1. When Professor Elliott asked if anyone knows what does the spectral theorem say, I was (and am still) surprised by that quite many people raised their hands and were confident that they know the spectral theorem. My (purely personal) opinion is that none of us even understands what the spectral theorem says after all! Stating the classical result doesn't really count...

2. SPECTRAL THEOREM

Let's recall the classical Spectral theorem, which gives a spectral decomposition of self-adjoint operators on Hilbert spaces. It states that any symmetric real matrix, or more generally hermitian complex matrix, is diagonalizable. We could interpret this as saying that the underlying Hilbert space $\mathcal{H} = \mathbb{C}^n$ can be written as a direct sum of eigenspaces

$$\mathcal{H} = \bigoplus_{\lambda \in \mathbb{R}_t} \mathcal{H}_\lambda$$

This simple but interesting result turns out to appear also in other contexts. We here focus on thinking of everything via the lens of algebraic geometry.

2.1. Algebraic Geometry Version.

2.1.1. We start with a simple observation. Let V be a vector space over k . Equipping a vector space V with an operator on itself is equivalent to making V a $k[x]$ -module. This is more or less apparent by the free-forgetful adjunction of k -algebras

$$\mathrm{Hom}_{\mathrm{Set}}(*, \mathbf{oblv}(\mathrm{End}(V))) \simeq \mathrm{Hom}_{\mathrm{CAlg}_k}(k[x], \mathrm{End}(V))$$

The upshot is, we should really be thinking of $(V, T : V \rightarrow V)$ as global sections to some quasicoherent sheaf on $\mathbb{A}^1$.

2.1.2. Like in the usual setting of linear algebra, let's restrict to the case when V is finitely generated, i.e. $\tilde{V}$ is a coherent sheaf. V "spreads out over $\mathbb{A}^1$ ", in the sense that we get a short exact sequence

$$0 \rightarrow V_{\mathrm{tor}} \rightarrow V \rightarrow V_{\mathrm{free}} \rightarrow 0$$

More particularly, in our case that V is a $k[x]$ -module, by the structure theorem for modules over PIDs, the sequence splits, and

$$V_{\mathrm{free}} = k[x]^{\oplus r}, \quad V_{\mathrm{tor}} = \bigoplus_{\lambda \in \mathrm{Spec}} V_\lambda$$

V_λ here is the subspace supported at λ , or really, by Jordan decomposition, we know that it is just $\bigoplus_i k[x]/(x - \lambda)^{l_i}$, i.e. generalized eigenspaces.

Remark 2.1.3. The way we should be thinking about this, is to mimic what we saw in the classical case. We should see V_{tor} as the discrete spectrum, and V_{free} as the continuous spectrum, as we shall see soon.

2.1.4. An λ -eigenspace (which is a subspace of V) in the $k[x]$ -module language could be understood as $\text{Hom}_{k[x]}(k_\lambda, V)$ (here $k_\lambda := k[x]/(x - \lambda)$), which geometrically is a section supported scheme theoretically at $\lambda \in \mathbb{A}^1$. Whereas the fibers $V \otimes_{k[x]} k_\lambda$ are naturally quotients, so they should be interpreted as some kind of “co-eigenvectors”.

2.1.5. Note that the “continuous spectrum”, which is the free part, cannot admit eigenvectors, so it really looks like sections of the trivial bundle. Whereas the discrete spectrum, looks like skyscrapers at points.

Example 2.1.6. Let’s say we take (V, T) as the $k[x]$ -module $k[x]$. This essentially tells us everything about the free case: There are no generalized eigenvectors, but many co-eigenvectors. For any $\lambda \in \mathbb{A}^1$, we get by taking projection $\delta_\lambda : V \rightarrow k_\lambda$, which is a distribution, meaning an element of $\text{Hom}(V, k)$

Example 2.1.7. How about discrete case? Let $V = L^2(\mathbb{R})$ and $T = x$, then V_λ the subspace of functions supported at λ is empty, because things like δ_λ are not L^2 . Say we consider “ $T^\vee$ ” = $\frac{d}{dx}$, things like $e^{i\lambda x}$ are not L^2 either.

2.1.8. The upshot is, when decomposing functions on $\mathbb{A}^1$, there is a sense in which it is a direct integral (not a direct sum) of functions on a point, but those functions do not live as subspaces. This is not something specific to analysis, because we just see it at the level of algebra.

2.2. Measurable version.

2.2.1. In the case we have a Hilbert space $\mathcal{H}$ and a self adjoint operator $\mathcal{A}$, von Neumann’s spectral theorem tells us that there is a projection valued measure $\pi_{\mathcal{A}}$ on $\mathcal{A}$ such that

$$\mathcal{A} = \int_{\mathbb{R}} x d\pi_{\mathcal{A}}$$

This also fits into our narrative before, in the sense that $\pi_{\mathcal{A}}$ could be thought as some kind of sheaf $\pi_{\mathcal{A}}$: For $U \subset \mathbb{R}$ measurable, take $\pi_{\mathcal{A}}(U) := \pi_{\mathcal{A}}(U)$

2.2.2. Note that even though we call it a “sheaf”, it is not really compatible with any sort of topology. But it still possess some kind of local-to-global property, via that

$$U \mapsto \langle w, \pi_{\mathcal{A}}(U)v \rangle$$

defines a $\mathbb{C}$ -valued measure on $\mathbb{R}$.

2.2.3. The upshot is, the Hilbert space $\mathcal{H}$ “sheafifies” over $\mathbb{R}$ such that $\mathcal{A}$ acts by the coordinate function x . Then the spectrum of $\mathcal{A}$ here could be interpreted as $\text{supp}(\pi_{\mathcal{A}})$. And in the case when $\mathcal{H}$ is finite dimensional, $\text{Spec}(\mathcal{A})$ is just a discrete set of points, so the integral is just a direct sum, we go back to decomposing $\mathcal{H}$ into eigenspaces!

3. FOURIER THEORY

We see above that whenever we make sense of our spectral objects, we have a concept of spectral decomposition in various contexts. In this section we shall embrace the philosophy and explore its applications in Fourier theory.

3.1. Baby Fourier Theory. Let’s fix our ground field to be $\mathbb{C}$ unless otherwise specified.

3.1.1. Given an abelian group representation i.e. a group homomorphism $\rho : G \rightarrow \text{Aut}(V)$ with G abelian, we get a family of operators $\{\rho(g)\}_{g \in G}$. Fourier theory in some sense, is the decomposition of V by these operators, which we shall see as an instance of Spectral Decomposition. We shall expand on this idea in the following.

3.1.2. Since we are doing spectral decomposition, the first question to ask is, what is our spectrum here? We would need to find our eigenvalues in this case. Note if $v \in V$ eigenvector, meaning for all $g \in G$,

$$g \cdot v = \chi(g)v$$

We see that $\chi : G \rightarrow \mathbb{C}^\times \subset \mathbb{C}$ is forced to be a group homomorphism, hence it is what is called a character of G . Therefore our spectrum is

$$\widehat{G} := \{\text{characters of } G\} = \text{Hom}(G, \mathbb{C}^\times)$$

3.1.3. At the moment, let's assume G to be finite for simplicity. Now let's repeat what we did earlier to carry out spectral decomposition:

First observe that a complex representation V gives rise to a monoid map from G to $\text{End}(V)$, then the adjunction

$$\text{Hom}_{\text{Mon}}(G, \mathbf{oblv}(\text{End}(V))) \simeq \text{Hom}_{\text{CAlg}_{\mathbb{C}}}(\mathbb{C}[G], \text{End}(V))$$

tells us that, we should be thinking of V as some kind of sheaf on $\text{Spec}(\mathbb{C}[G], *)$. We shall make these precise in a moment.

3.1.4. *Digression on $\mathbb{C}[G]$.* I am sure everyone has encountered $\mathbb{C}[G]$ and the adjunction before in their first course of algebra. But there's something more to be said:

Recall that $\mathbb{C}[G] = \{\sum_{g \in G} f(g)g\}$, for which $f : G \rightarrow \mathbb{C}$ is some function. Even though it does not matter when G is finite, it is better to interpret f as a measure, instead of a function. Perhaps an example that might convince you is the element $\delta_g = 1 \cdot g$, where f here would be 1 at g and 0 otherwise, does this ring a bell?

Recall that the algebra structure of $\mathbb{C}[G]$ is given by

$$f_1 * f_2 = \left(\sum_g f_1(g)g \right) * \left(\sum_h f_2(h)h \right) = \sum_k \left(\sum_{gh=k} f_1(g)f_2(h) \right) k$$

This convolution product has an alternative description (that turns out to generalize quite well): Given $\mu : G \times G \rightarrow G$ and $\pi_1, \pi_2 : G \times G \rightarrow G$ the canonical projections, then

$$f_1 * f_2 = \mu_*(f_1 \boxtimes f_2) = \mu_*(\pi_1^* f_1 \pi_2^* f_2) = \int_{\mu} f_1 \boxtimes f_2$$

3.1.5. Now back to $\text{Spec}(\mathbb{C}[G], *)$. Note that $\mathbb{C}[G]$ is a commutative algebra under $*$, so this indeed makes sense. It turns out that $\text{Spec}(\mathbb{C}[G], *) = \widehat{G}$, because a (closed) point of a spectrum is essentially $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{C}[G])$, hence a map $\mathbb{C}[G] \rightarrow \mathbb{C}$, which determines a character, and vice versa. Hence we indeed have

$$(\mathbb{C}[G], *) \simeq (\mathcal{O}(\widehat{G}), \cdot)$$

where characters on the left χ_t are exchanged with δ -functions δ_t on the right. And translation by g on the left corresponds to multiplication by some $\hat{g}$.

The upshot is, this could indeed be interpreted as some version of Fourier transform: if f is a function on G , by expressing it via the basis of characters, then

$$f = \sum_{t \in \widehat{G}} \hat{f}(t) \cdot \chi_t$$

The above correspondence tells us that indeed f and $\hat{f}$ could be recovered from each other, and

$$f * (-) = \hat{f} \cdot (-)$$

3.1.6. There is actually a piece of data that we missed the entire time: $\widehat{G}$ is actually an abelian group via pointwise multiplication. This will indeed be called the dual group of G , for the reason that:

$$G \rightarrow \widehat{\widehat{G}}, \quad g \mapsto \{\chi \mapsto \chi(g)\}$$

is an isomorphism.

There is a way that makes it more symmetrical, as we did not really provide an inverse map from the double dual earlier. Observe that there is the universal character $\chi(-, -) : G \times \widehat{G} \rightarrow \mathbb{C}^\times$ such that $\chi(g, t) = \chi_t(g)$ (which you should think as the multiplicative kernel in functional analysis). Then the Fourier transform of f could be described by this pull-push procedure:

$$\begin{array}{ccc} & G \times \widehat{G} & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ G & & \widehat{G} \end{array}$$

$$f \mapsto (\pi_2)_*((\pi_1^* f) \cdot \chi)$$

Hence

$$\hat{f}(t) = \sum_g f(g) \bar{\chi}(g, t), \quad f(g) = \sum_t \hat{f}(t) \chi(g, t)$$

So we have simultaneously diagonalized the action of G on the space of functions on G .

The upshot after all of this is: given a G -representation V , hence a $(\mathbb{C}[G], *)$ -action on V , with respect to the spectrum $\widehat{G}$, V spectrally decomposes as

$$V = \bigoplus_{t \in \widehat{G}} V_{\chi_t}$$

where G acts by the eigenvalue χ_t on the subspace V_{χ_t} .

3.2. Pontrjagin Duality. Here we shall go through the exact same story, but we replace G by a locally compact abelian group.

Example 3.2.1 (Non-finite examples of LCAs). Just to make sure we are not producing a redundant theory, these objects exist (Epistemically? Ontologically? whatever you like): $\mathbb{Z}, U(1), \mathbb{R}, \mathbb{Q}_p, \mathbb{Q}_p^*, \dots$

3.2.2. We first define our dual here (the Pontrjagin dual), which is the collection of unitary characters

$$\widehat{G} := \text{Hom}_{\text{TopGrp}}(G, U(1))$$

We endow G with a Haar measure (in place of the counting measure previously). Our group algebra will be the C^* -algebra $L^1(G)$, which has the convolution product:

$$f_1 * f_2 = \int_{g \in G} f_1(h) f_2(gh^{-1}) dg$$

Just as before the convolution satisfies a universal property in the world of C^* -algebras:

Given a representation $G \rightarrow \text{End}(V)$, it corresponds to a C^* algebra map

$$(L^1(G), *) \rightarrow \text{End}(V)$$

The spectrum is the Gelfand spectrum (this is what inspired Grothendieck to use the name spectrum for the collection of prime ideals!),

$$\text{mSpec}(L^1(G), *) \simeq \widehat{G}$$

This is a version of the Fourier transform that tells us

$$(L^1(G), *) \simeq (C_v(\widehat{G}), \cdot)$$

where C_v denotes functions vanishing at ∞ .

3.2.3. Again, we have that

$$G \rightarrow \widehat{\widehat{G}}$$

is an isomorphism, which can be written in the symmetric way via:

$$f \mapsto (\pi_2)_*(\pi_1^* f \cdot \chi)$$

In fact this Fourier transform operation works for any function or distribution. In the case of L^2 functions, by the Plancherel theorem

$$(L^2(G), *) \simeq (L^2(\widehat{G}), \cdot)$$

Any of these notions of what we call a Fourier transform have the same features:

- (1) Translation by a group element becomes pointwise multiplication
- (2) Convolution becomes pointwise multiplication
- (3) Characters correspond to points

Example 3.2.4 (Fourier Series: $G = U(1)$). In this case

$$\widehat{G} = \text{Hom}_{\text{TopGrp}}(U(1), U(1)) \simeq \mathbb{Z}$$

where

$$[x \mapsto e^{2\pi i n x}] \leftarrow n$$

Then Fourier transform establishes

$$(L^2(U(1)), *) \simeq (L^2(\mathbb{Z}), \cdot) = \ell^2$$

There is a universal character $\chi(x, n) = e^{2\pi i n x}$ that lives over $U(1) \times \mathbb{Z}$. And since unitary characters $\mathbb{Z} \rightarrow U(1)$ is determined by where 1 goes, $\widehat{\mathbb{Z}} = U(1)$.

3.2.5. So far, we still have not really showed that our approach recovers the usual Fourier transform, so let's do that here: Replace $U(1)$ by a compact torus T , i.e.

$$T = \mathbb{R}^d / \Lambda = \Lambda \otimes_{\mathbb{Z}} U(1)$$

where $\Lambda \subset \mathbb{Z}^d$ is the full rank lattice. Then the dual of T is just the dual lattice $\Lambda^\vee$.

For $G = \mathbb{R}_x$ (we take x to be the coordinate), its dual $\widehat{G}$ is isomorphic to itself, which we shall denote by $\mathbb{R}_t$.

Remark 3.2.6. In fact, the dual of any finite dimensional real vector space is just its linear dual, hence isomorphic to itself.

3.2.7. Repeat the same procedure, with the universal character

$$\chi(x, t) = e^{2\pi i \langle x, t \rangle}$$

where $\langle \cdot, \cdot \rangle$ the pairing between vector space and its dual, we get

$$f(x) = \int_{\mathbb{R}} \hat{f}(x) e^{2\pi i x t} dt$$

Here, differentiation $\frac{d}{dx}$ corresponds to multiplication by t on the other side, so $\frac{d}{dx}$ should be thought as some infinitesimal version of group translation. This recovers Fourier transform.

3.2.8. In general, if G is a Lie group with an abelian Lie algebra $\mathfrak{g}$, we have a map from $\mathfrak{g}$ to the vector fields on G

$$\mathfrak{g} \rightarrow \text{Vect}(G) \subset \text{Diff}(G)$$

By the adjunction

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, A) \simeq \text{Hom}_{\text{AssocAlg}}(U(\mathfrak{g}), A)$$

We know that $U(\mathfrak{g})$ acts on $C^\infty(G)$ as differential operators. And $U(\mathfrak{g}) = \text{Sym}(\mathfrak{g})$ as $\mathfrak{g}$ is abelian, hence is a commutative algebra. So we can spectrally decompose / sheafify over $\text{Spec } U(\mathfrak{g}) = \mathfrak{g}^*$.

For example when $G = \mathbb{R}_x$, $\mathfrak{g}^* = \mathbb{R}_t$ could be identified with $\text{Spec } \mathbb{R}[t]$ with $t = \frac{d}{dx}$, which explains why differentiation becomes multiplication by t .

Example 3.2.9 (Quantum Mechanics). Quantum mechanics is where we often encounter the idea of Fourier transform. Recall that in classical mechanics, for a particle moving on a manifold M , the phase space is given by T^*M , with local coordinates q , for position, and p for momentum in the fiber direction. They generate the algebra of observables.

Whereas in quantum mechanics, the phase space is instead $L^2(M)$, so we only have functions on half of the variables, namely the positions. The observables include differential operators on M , i.e. functions on M and tangent vectors, with $p_j = i \frac{d}{dq_j}$. Unlike classical mechanics, if $\xi \in TM$ and $f : M \rightarrow \mathbb{R}$, there is a commutator

$$\xi f = f\xi + \hbar f'$$

where $\hbar$ is the Planck's constant.

Here “states look like the square root of the observables” because only half of the observables classically are left to quantize. Here the position operators are diagonalized, while the momentum operators act as derivatives. We can do the opposite instead: say if $M = \mathbb{R}^n$, then we have a natural basis of commuting vector fields, so by Fourier transform:

$$L^2(\mathbb{R}_q^n) \simeq L^2(\mathbb{R}_p^n)$$

Physically this is saying one identifies the quantum mechanics on G with that on $\widehat{G}$. Note that this could only be seen at the non-classical level. This can be seen as a very simple version of abelian duality or 1D-mirror symmetry.

3.3. Fourier Theory in Algebraic Geometry. Here we discuss a version of what we have done in the realm of Algebraic Geometry. This is known as Cartier duality.

3.3.1. We first give a crash course on algebraic groups. We shall embrace the functor of points philosophy here: An algebraic group is just a group object in the category of varieties. i.e.

$$G(R) = \text{Hom}_{\text{CAlg}_k}(\text{Spec } R, G) \text{ is an abelian group}$$

Let's see some examples:

- (1) $G = \mathbb{G}_a$, $\mathbb{G}_a(R) := R$. Note that this essentially is $\mathbb{A}^1 = \text{Spec } k[x]$, but we commonly use the notation $\mathbb{G}_a$ if we want to specify it is an algebraic group
- (2) $G = \mathbb{G}_m$, $\mathbb{G}_m(R) = R^\times$. This is essentially $\mathbb{A}^1 \setminus \{0\} = \text{Spec } k[t, t^{-1}]$.
- (3) Constant functors valued in an abelian group

3.3.2. Let G be an algebraic group, then its Cartier dual is given by

$$\widehat{G} := \text{Hom}_{\text{AlgGrp}}(G, \mathbb{G}_m)$$

The reason for $\mathbb{G}_m$ is because it is just $\text{Aut}(k)$. Some examples are:

- (1) $\widehat{\mathbb{Z}/n} = \mu_n$
- (2) $\widehat{\mathbb{Z}} = \mathbb{G}_m$
- (3) $\widehat{T} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m$

3.3.3. Say we restrict to G finite, then if we consider the functions $\mathcal{O}(G)$ on G , then

$$\widehat{G} = \text{Spec}(\mathcal{O}(G))^*$$

$\mathcal{O}(G)$ also have a Hopf algebra structure, whose comultiplication is induced by the multiplication of G . More precisely, it is a finite dimensional commutative and cocommutative Hopf algebra.

Example 3.3.4 (Fourier Transform). Fourier Transform in this case could be interpreted by the result

$$\widehat{\mathbb{G}_a} = \mathbb{G}_a^\wedge$$

where $\mathbb{G}_a^\wedge$ denotes the formal completion given by

$$\mathbb{G}_a^\wedge = \varinjlim_n \text{Spec } k[t]/(t^n)$$

In general, the Cartier dual of a vector space V is the formal completion of its linear dual:

$$\widehat{V} = (V^*)^\wedge$$

The character of $\mathbb{G}_a$ should be something like

$$e^{xt} = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!}$$

But we would like this to be finite, hence, we require t to be nilpotent. Therefore when t is “infinitesimally small”, the expression $e^{\langle x, t \rangle}$ makes sense, with $x \in V$ and $t \in V^*$, and forms a character for V .

Similarly, the dual of the formal completion gives back $\mathbb{G}_a$:

$$\mathbb{G}_a = \widehat{\mathbb{G}_a^\wedge}$$

3.3.5. Recall that in Fourier duality, we have

$$(\mathrm{Fun}(G), *) \simeq (\mathrm{Fun}(\widehat{G}), \cdot)$$

and $\mathrm{Rep}(G)$ is spectrally decomposed over $\widehat{G}$. In the case of algebraic geometry, we have the following identifications:

$$\mathrm{Rep}(G) \simeq \mathrm{comod}_{\mathcal{O}(G)} \simeq \mathrm{comod}_{\mathcal{O}(\widehat{G})} \simeq \mathrm{QCoh}(\widehat{G})$$

The reason being, if one has the G action $G \times V \rightarrow V$, then passing to functions corresponds to $V \rightarrow \mathcal{O}(G) \otimes V$. Let’s see some specific cases

- (1) ($G = \mathbb{Z}$): Under the above identification:

$$\mathrm{Rep}(\mathbb{Z}) = \mathrm{Mod}_{k[t, t^{-1}]} = \mathrm{comod}_{\mathcal{O}(\mathbb{G}_m)} = \mathrm{QCoh}(\mathbb{G}_m)$$

Recall from a discussion earlier about identifying a vector space with endomorphism as a quasicoherent sheaf on $\mathbb{A}^1$. Essentially if the endomorphism is in fact invertible, i.e. an automorphism, we could identify it as a quasicoherent sheaf on $\mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$, so it falls into this discussion.

- (2) ($G = \mathbb{G}_m$) Under the identification:

$$\mathrm{Rep}(\mathbb{G}_m) = \mathrm{GrZVect} = \mathrm{QCoh}(\mathbb{Z})$$

Here $V \simeq \bigoplus_{n \in \mathbb{Z}} V_n$ and $z \in \mathbb{G}_m$ acts on V_n by z^n .

- (3) ($G = \mathrm{Pic} M^3$) where M^3 is a compact oriented 3-manifold, and $\mathrm{Pic} M^3$ denotes the complex line bundles on M^3 up to isomorphism. Topologically we know that we could identify

$$\mathrm{Pic} M^3 \simeq \mathrm{Maps}(M^3, BU(1))$$

where $BU(1)$ is the classifying space of $U(1)$, which has a more concrete description up to homotopy

$$BU(1) \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z}; 2)$$

where K denotes the Eilenberg-MacLane space. We can further identify the mapping space as

$$\Lambda = H^2(M^3; \mathbb{Z})$$

via taking the first chern class $c_1(\mathcal{L})$. Then the Cartier dual gives us

$$\widehat{G} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m = \mathrm{Hom}(H^1(M^3; \mathbb{Z}), \mathbb{G}_m) = \mathrm{Hom}(\pi_1(M^3), \mathbb{G}_m)$$

via the universal coefficient theorem and the Hurewicz theorem. The object on the right should be familiar to all geometers, it is the flat $\mathbb{C}^\times$ -bundles on M^3 , usually denoted $\mathrm{Loc}_{\mathbb{C}^\times} M^3$.

This kind of matches our intuition that G which is something that looks like $\mathbb{Z}^n$ has its Cartier dual something that looks like $(\mathbb{C}^\times)^n$.

We can replace $U(1)$ with $T \simeq U(1)^n$, then we get

$$\mathrm{Bun}_T(M) \leftrightarrow \mathrm{Loc}_{T^\vee}(M)$$

duality between torus bundles and flat $T^\vee$ -bundles.

The fact that we get more geometric objects on both sides suggest that this kind of duality result is itself interesting to study, not just a consequence of Cartier duality. In fact these arises a lot in electric-magnetic duality! It even has an application to arithmetic: consider a number field as a 3-manifold (this is the Arithmetic Topology Dictionary, which is itself an interesting thing to study, but we don't have space to expand on this.), you get usual class field theory!

3.4. Pontrjagin-Poincare Duality. In this section we outline particular interesting examples of Pontrjagin / Fourier duality, that involves topology.

3.4.1. Let's start with the set-up. Consider M^n a compact oriented n -manifold. We study our duality on cohomology groups of M . Observe that

$$\widehat{H}^i(M^3; G) \simeq H^{n-i}(M^3; \widehat{G})$$

because there is a nondegenerate pairing:

$$H^i(M; G) \otimes H^{n-i}(M; \widehat{G}) \rightarrow H^n(M; G \otimes \widehat{G}) \rightarrow H^n(M; U(1)) \rightarrow U(1)$$

This is related to a fairly general abstract version of Pontrjagin duality called Brown-Comenetz duality (which I would like to write about in the second essay, as apparently it admits a condensed formulated version).

Let's see some examples in each dimension. An idea to keep in mind is, we should really be thinking both sides to be corresponding to different "field theories", the side we start with being the "A-side" and the dual being the "B-side".

Example 3.4.2 (dimension = 1). On the "A-side", we shall get $\text{Maps}(*, \mathbb{Z}) = \mathbb{Z}$, whereas on the "B-side", we get $\text{Maps}(*, \mathbb{C}^\times) = \mathbb{C}^\times$, which says "quantum mechanics of $\mathbb{Z}$ " is identified with that of $\mathbb{C}^\times$, not extremely interesting.

Example 3.4.3 (dimension = 2). On the "A-side", we can consider

$$H^0(S^1; \mathbb{Z}) = \pi_0(\text{Maps}(S^1, \mathbb{Z})) = [S^1, \mathbb{Z}]$$

whereas on the "B-side", we will get

$$H^1(S^1; \mathbb{C}^\times) = |\text{Loc}_{\mathbb{C}^\times}(S^1)|$$

the underlying space of flat $\mathbb{C}^\times$ -bundles over S^1 . Even though both sides again just look like $\mathbb{Z}$ and $\mathbb{C}^\times$ respectively, we actually do see that this has to do with some kind of $U(1)$ -gauge theory, i.e. $U(1)$ -local systems / principal bundles, by thinking 1 dimension higher.

3.4.4. *Digression on Local Systems.* Let G be a discrete group. Recall that a G -local system (on space X) is just a sheaf which is locally isomorphic to the constant sheaf $\underline{G}$, in which the gluing maps are identified with just G acting on itself by multiplication. The particular thing to notice is that, we do not actually use anything about the topology or geometry of G , but just the underlying set. Up to isomorphism, we know that it is determined by maps

$$\pi_1(X) \rightarrow G$$

And in fact we could identify its isomorphism classes with $H^1(X; G)$.

Example 3.4.5 (dimension = 2). Another more interesting example is to consider $H^1(S^1, \mathbb{Z})$ on the "A-side", then

$$H^1(S^1, \mathbb{Z}) = [S^1, B\mathbb{Z}]$$

where

$$B\mathbb{Z} = K(\mathbb{Z}; 1) = S^1$$

Hence just maps into S^1 . On the other hand, the "B-side" $H^0(S^1; \mathbb{C}^\times)$ is just maps into $\mathbb{C}^\times$ (with the discrete topology). Here both sides could be seen as something called a scalar field, and this duality is a very special example of what is called T -duality, or mirror symmetry.

3.4.6. Like what we did before, one can try replace $\mathbb{Z}$ with a torus. Using a similar calculation, we would get a correspondence between maps into the torus on the “A-side” and maps into the dual torus on the “B-side”.

One can proceed similarly, and we should get the following:

Dimension	A-side	B-side
1	$\text{Map}(\text{pt}, \mathbb{Z}) = \mathbb{Z}$	$\text{Map}(\text{pt}, \mathbb{C}^\times) = \mathbb{C}^\times$
2	$H^0(S^1, \mathbb{Z})$ $H^1(S^1, \mathbb{Z}) = [S^1, S^1]$	$H^1(S^1, \mathbb{C}^\times) = \text{Loc}_{\mathbb{C}^\times}(S^1)$ $H^0(S^1, \mathbb{C}^\times)$
3	$H^1(\Sigma, \mathbb{Z}) = [\Sigma, S^1]$ $H^2(\Sigma, \mathbb{Z}) = [\Sigma, BU(1)]$	$H^1(\Sigma, \mathbb{C}^\times) = \text{Loc}_{\mathbb{C}^\times}(\Sigma) $ $H^0(\Sigma, \mathbb{C}^\times)$
4	$H^2(M^3; \mathbb{Z}) = \{\text{lines bundles on } M^3 / \sim\}$	$H^1(M^3, \mathbb{C}^\times) = \text{Loc}_{\mathbb{C}^\times} M^3 $

There are things to be said about electric-magnetic duality here, but to stay relevant to the theme of our essay, let's stop here.

4. CATEGORIFICATION

In this section, we will categorify some of the things we saw in earlier sections.

4.1. Tannakian Reconstruction.

4.1.1. Recall that quasi-coherent sheaves on a scheme (or more generally a prestack) admits a (2)-limit description (I personally just never use the terminology 2-limit and just take the limit in the ∞ -sense but I guess we are working classically):

$$\text{QCoh}(X) = \lim_{\text{Spec } R \rightarrow X} \text{Mod}_R$$

Some useful things to note:

- (1) $\text{QCoh}(X)$ is symmetric monoidal with tensor product $\otimes_{\mathcal{O}_X}$ and unit $\mathcal{O}_X$
- (2) if $f : X \rightarrow Y$ is a map of schemes, there is a pullback functor $f^* : \text{QCoh}(Y) \rightarrow \text{QCoh}(X)$ that is symmetric monoidal.
- (3) $\text{QCoh}(X)$ is the universal object that satisfies the property that: if there is a map $f : \text{Spec } R \rightarrow X$, then there is a functor $f^* : \text{QCoh}(X) \rightarrow \text{Mod}_R$ that is symmetric monoidal

Note that then we have

$$\text{QCoh} : \text{Sch} \rightarrow (\otimes \text{Cat})^{op}$$

This functor into symmetric monoidal categories (or $\otimes$ -categories) could be seen as the categorification of

$$\mathcal{O} : \text{Sch} \rightarrow \text{CAlg}^{op}$$

We could indeed enhance QCoh to have its source to be the category of stacks:

$$\text{QCoh} : \text{Stk} \rightarrow (\otimes \text{Cat})^{op}$$

The functor QCoh gives a geometric source of $\otimes$ -categories. A natural question to ask is if we have a right adjoint

$$\text{Spec} : (\otimes \text{Cat})^{op} \rightarrow \text{Stk}$$

that would be our analogue of the usual Spec . The idea is that it boils down to considering affines, so we want

$$\text{Hom}_{\text{Stk}}(\text{Spec } R, \text{Spec } \mathcal{C}) \simeq \text{Hom}_{\otimes \text{Cat}}(\mathcal{C}, \text{Mod}_R)$$

So we can force this by defining $\text{Spec } \mathcal{C}(R) = \text{Hom}_{\otimes \text{Cat}}(\mathcal{C}, \text{Mod}_R)$.

By considering the unit of the adjunction, we get a canonical map

$$X \rightarrow \text{Spec}(\text{QCoh}(X))$$

which should be seen as a 1-affinization / Tannakianization of X (i.e. X 's best affine approximation).

Example 4.1.2 ($\mathcal{C} = \text{Mod}_S$). Let's compute and see if our categorification recovers our old notion. Our claim is every $\otimes$ -functor $\text{Mod}_S \rightarrow \text{Mod}_R$ corresponds to the ring map $S \rightarrow R$. First note that

$$\text{Hom}_{\otimes\text{Cat}}(\mathcal{C}, \text{Mod}_R) \rightarrow \text{Hom}(\text{End}(1), R)$$

as $1 \mapsto R$. In the case where $\mathcal{C} = \text{Mod}_S$, we know every element of Mod_S is generated by S , which is the unit 1, so indeed

$$\text{Hom}_{\otimes\text{Cat}}(\text{Mod}_S, \text{Mod}_R) \xrightarrow{\sim} \text{Hom}(S, R)$$

4.1.3. So what is up with this categorification business? The upshot is the affinization map via $\mathcal{O}$ could not recover our scheme usually because we don't have enough functions. By lifting 1 categorical level up, things change:

Theorem 4.1.4. *If X is a geometric stack, then $X \xrightarrow{\sim} \text{Spec } QC(X)$ is an isomorphism*

Remark 4.1.5. You might think I am very particular about writing down the directions of the arrow despite having isomorphisms. Once somebody told me that one should actually know where the arrow goes because it tells you which direction of the correspondence is easier, and where you get your data. If you don't specify it, it means both sides produce equally interesting information. And clearly here, this arrow is easier because it is just the unit of our adjunction.

Example 4.1.6 ($\mathcal{C} = \text{Rep}(G)$). Observe that we have a fiber functor, i.e. a faithful $\otimes$ -functor

$$\text{Rep}_k(G) \rightarrow \text{Vect}_k$$

In this case, we have a faithfully flat map $* \rightarrow \text{Spec } \text{Rep}_k(G)$. When G is an affine algebraic group, $\text{Spec } \text{Rep}_k(G) \simeq BG = */G$.

In general, if we have $\mathcal{C}$ with a fiber functor,

$$\text{Spec } \mathcal{C} \simeq BG^{\text{proalg}}$$

where G^{proalg} is the pro-algebraic completion of G .

Example 4.1.7 ($\mathcal{C} = \text{Gr}_{\mathbb{Z}} \text{Vect}$). Here the tensor product is given by

$$(M \otimes N)_n = \bigoplus_{i=1}^{n-1} M_i \otimes N_{n-i}$$

To calculate $\text{Spec}(\text{Gr}_{\mathbb{Z}} \text{Vect})$, Note that we have a fiber functor

$$\text{Gr}_{\mathbb{Z}} \text{Vect} \xrightarrow{F} \text{Vect}$$

Hence $\text{Gr}_{\mathbb{Z}} \text{Vect} = \text{Rep}(G)$ for some G , here that is $G = \mathbb{G}_m$, then we get the equivalence we had earlier

$$\text{Gr}_{\mathbb{Z}} \text{Vect} \simeq \text{Rep}(\mathbb{G}_m) \simeq \text{QCoh}(B\mathbb{G}_m)$$

So we have a categorical description of the Fourier duality between $\mathbb{Z}$ and $\mathbb{G}_m$.

4.2. Fourier-Mukai Theory. Here we carry out similar procedures to what we did earlier in Cartier duality theory, but categorically.

4.2.1. Instead of the multiplication map $\mu : G \times G \rightarrow G$, we replace it with the pushforward map induced by μ ,

$$\mu_* : \text{Shv}(G) \times \text{Shv}(G) \rightarrow \text{Shv}(G), \quad (\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{F} * \mathcal{G}$$

where $*$ is the convolution product. And we replace the Cartier dual here (or what we will refer as, the Fourier-Mukai dual / 1-shifted Cartier dual) by

$$\widehat{G}[1] = \text{Spec}(\text{Shv}(G), *)$$

so here $\text{Shv}(G)$ is our categorified group algebra.

Remark 4.2.2. We are deliberately unclear about what is Shv here. Because topologically, it could be local systems, geometrically, it could be QCoh ...

Example 4.2.3 ($G = \mathbb{Z}$). Let's play with our favorite example. Essentially $\mathrm{Shv}(\mathbb{Z}) = \mathrm{Gr}_{\mathbb{Z}} \mathrm{Vect}$, then by our previous calculation, we get $B\mathbb{G}_m$.

Example 4.2.4 (Ordinary Cartier Duality). Consider G finite group scheme (such that Cartier duality works). Recall usual Cartier Duality says

$$\widehat{G} = \mathrm{Spec}(\mathbb{C}[G], *)$$

Passing to modules, we get

$$\mathrm{Rep}(G) \simeq \mathrm{QCoh}(\widehat{G})$$

But we also know that $\mathrm{Rep}(G)$ could be understood as $\mathrm{QCoh}(BG)$ from geometry, therefore we get

$$\mathrm{QCoh}(BG) \simeq \mathrm{QCoh}(\widehat{G})$$

So here we exchange G with the classifying space of its ordinary Cartier dual, which is why we called the Fourier-Mukai dual the 1-shifted Cartier dual. This is exactly what we saw in the above example.

4.2.5. Let's study the Fourier-Mukai dual in more detail: Note that the k -points of $\widehat{G}[1]$ is

$$(\mathrm{QCoh}(G), *) \rightarrow (\mathrm{Vect}_k, \otimes)$$

We really should be thinking of Vect_k as $\mathrm{End}(\mathrm{Vect}_k)$, because recall a 1-dimensional group representation is just

$$\mathbb{C}[G] \rightarrow \mathrm{End}(k) \simeq k$$

This map is supposed to be a “categorical 1-dimensional group representation”. Hence we can restrict to invertible elements to get

$$G \rightarrow \mathrm{Aut}(\mathrm{Vect}_k)$$

Note that automorphisms are just given by tensoring with lines, so $\mathrm{Aut}(\mathrm{Vect}_k) \simeq (\mathrm{lines}, \otimes) \simeq B\mathbb{G}_m$. Hence each k -point of $\widehat{G}[1]$ is just given by the map

$$G \rightarrow B\mathbb{G}_m$$

Recall that $\mathrm{Spec}(\mathbb{C}[G], *)$ can be given by characters $G \rightarrow \mathbb{G}_m \simeq \mathrm{Aut}(k)$. So it aligns with the classical philosophy! We could interpret the k -points as “categorical characters / character sheaves”.

More precisely, the map $G \rightarrow B\mathbb{G}_m$ is actually a line bundle $\mathcal{L}$ over G that is multiplicative, hence

$$\widehat{G}[1] = \mathrm{Ext}_{\mathrm{AbGrp}}^1(G, \mathbb{G}_m)$$

Remark 4.2.6. The shift here is a literal shift in the derived category of sheaves of abelian groups on CAlg^{op} , as commutative group schemes / stacks live naturally in that category.

4.2.7. Now after setting up, we can do the categorified Fourier Transform:

For a reasonably nice G (which we will be unclear about as it is not too important to illustrate the ideas), we have the Fourier-Mukai transform,

$$\mathrm{QCoh}(G) \rightarrow \mathrm{QCoh}(\widehat{G}[1]), \quad \mathcal{F} \mapsto (\pi_2)_*(\pi_1^* \mathcal{F} \otimes \mathcal{L})$$

Remark 4.2.8. The original example that introduces this considers $G = A$ to be an abelian variety, the dual abelian variety A^* turns out to be $\widehat{A}[1]$, hence we get

$$\mathrm{QCoh}(A) \rightarrow \mathrm{QCoh}(A^*)$$

Here everything is derived (side note: Somebody did told me that everything should be assumed to be derived at the start, because that is the natural setting for everything). Note that we need to work in the derived world because the pushforward functor is not exact in general.

4.2.9. Let's call back one of our themes, spectral decomposition, to end this section. If we have $(\mathcal{C}, *)$ acting on $\mathcal{M}$, we get a module category for $\mathrm{QCoh}(\mathrm{Spec}(\mathcal{C}))$. This gives rise to a sheaf of categories over $\mathrm{Spec}(\mathcal{C})$. And by Gaitsgory's 1-affineness, this is an equivalence between modules over $\mathrm{QCoh}(\mathrm{Spec}(\mathcal{C}))$ and sheaves of categories over $\mathrm{Spec}(\mathcal{C})$. This, could be interpreted as some kind of categorified Spectral decomposition.

5. GLIMPSE INTO GEOMETRIC LANGLANDS

5.0.1. We shall end this essay with something that I do not know anything about and is eager to learn about. The idea is that, the spectral theorem, and its variants, could be seen as instances of the Geometric Langlands Correspondence.

5.0.2. Recall the algebraic geometry version of the Spectral theorem that we discussed, which basically states that

$$\{\text{Vector spaces } V \text{ over } \mathbb{C} \text{ with an endomorphism } H\} \leftrightarrow \{\text{Quasicoherent sheaves on } \mathbb{A}_{\mathbb{C}}^1\}$$

The algebraic spectral theorem allows us to identify (V, H) , where H is an invertible operator, to modules over $\mathbb{C}[t, t^{-1}]$, hence quasicoherent sheaves over $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\} = \mathbb{C}_t^{\times}$.

The upshot is that, the collection of (V, H) could also be identified as $\text{Rep}(\mathbb{Z})$, and $\mathbb{Z}$ could indeed be identified as $\pi_1(\mathbb{C}_z^{\times})$, the fundamental group of the punctured z -line. This by the Riemann-Hilbert Correspondence, describes the monodromy of a local system of vector spaces on $\mathbb{C}_z^{\times}$.

5.0.3. Since the simplest local systems are rank one families $\mathcal{L}_t$ where the monodromy is just multiplication by a nonzero scalar t , the spectral theorem can be reinterpreted as the statement that any local system $L \in \text{Loc}(\mathbb{C}_z^{\times})$ could be realized as a superposition (direct integral) of the $\mathcal{L}_t$. So we have an equivalence, the Betti-Mellin transform for local systems

$$\text{Loc}(\mathbb{C}_z^{\times}) \leftrightarrow \text{QCoh}(\mathbb{C}_t^{\times})$$

5.0.4. The same spectral decomposition appears in the study of differential equations, we shall not dive into detail here. But at the end we result in the de Rham version of the Betti-Mellin transform

$$\text{D-mod}(\mathbb{C}_z^{\times}) = \{\text{alg. systems of diff. eqts. on } \mathbb{C}_z^{\times}\} \leftrightarrow \text{QCoh}(\mathbb{C}_s/\mathbb{Z})$$

5.0.5. What turns out is, these are special cases of the geometric Langlands correspondence. In the case of the Mellin transform, we get to fix a group and a Riemann surface, which in this case is just $G = GL_1$ and $\mathbb{C}_z^{\times}$, or more officially the Riemann sphere $\mathbb{P}^1$ with tame ramification at $0, \infty$.

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